Limit cycles in second order systems through sliding surface design

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Abstract. In this paper it is designed a sliding mode controller to generate a limit cycle in a nonlinear second order system using only one control input, it is considered that the systems is affected by a discontinuous function and a periodic disturbance. The proposed controller does not need an exact knowledge of the discontinuous function and disturbances; it only needs an upper bound of their magnitudes. It is proved that the limit cycle is reached in finite time. It is worth mention that the designed controller application is straightforward to a second order mass-spring-damper system. The performance of the proposed controller is illustrated in a numerical simulation.

Keywords: Limit cycles, sliding surfaces, second-order systems

1 Introduction

A recent work of generation of stable limit cycles with prescribed frequency and amplitude via polinomial feedback is presented in [Knoll and Robenack(2012)], also another previous work of generation of limit cycles in linear systems without perturbations is given by [Bacciotti et al.(1996)Bacciotti, Mazzi, and Sabatini].

This paper is about the generation of limit cycles through sliding mode technique, the main feature of this class of controllers is to allow the sliding mode to occur on a prescribed switching surface, so that the system is governed by the sliding equation only, and remains insensitive to a class of disturbances and parameter variations [Utkin(1978)]. This control method has been successfully tested for motion control of robotic manipulators, see [Sabanovic(2008)] and references therein. Besides, a previous work of sliding-mode control can be found in [Rascon et al.(2012)Rascon, Alvarez, and Aguilar].

The problem addressed in the present paper is the generation of limit cycles in nonlinear second order systems, the limit cycle could have prescribed frequency and amplitude as mentioned later, moreover the trajectories reach the limit cycle in finite time in spite of perturbations and nonlinear discontinuous phenomena, in order to achieve the control objective is necessary to know the upper bounds of the perturbations and the nonlinear term affecting the system.

The rest of the paper is outlined as follows: In Section II we describe the second order nonlinear system with perturbations. The control design and sliding surface proposed are presented in Section III. Section IV presents stability analysis. Section V presents an academic example performed with MATLAB®. Finally Section V includes some final comments.

2 Problem statement

Basically, the proposed problem it is about to design a discontinuous controller for the dynamic system

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= f(x) + g(x)u + w(t) \end{aligned} \tag{1}$$

capable of guaranteeing that the trajectories of (1) will converge to a limit cycle in finite time. Consider that $x = [x_1, x_2]^T$ is the state vector, $u \in \mathbb{R}$ is the control input, the nominal dynamics of the system are giving by $f(x) = -Ax_1 - Bx_2 - \alpha\,\mathrm{sign}(x_2)$, $g(x) \in \mathbb{R}$ is a well known function and $w(t)$ is a matched uncertainty/perturbation with an upper bound M that is assumed known a priori, so it satisfies

$$\sup_t |w(t)| \leq M, \qquad M > 0 \tag{2}$$

for all t and some constant $M > 0$. Also, the term α is considered with a level denoted as $\alpha > 0$. The $\mathrm{sign}(x_2)$ denotes the signum function defined as

$$\mathrm{sign}(x_2) = \begin{cases} 1 & x_2 > 0 \\ [-1, 1] & x_2 = 0 \\ -1 & x_2 < 0. \end{cases} \tag{3}$$

Then, for a constant force input $u = \bar{u}$ and zero disturbance ($w(t) = 0$), the system (1) has the equilibrium point $\bar{x}_2 = 0$ and $\bar{x}_1 \in [(\bar{u} - \alpha)/A, (\bar{u} + \alpha)/A]$.

3 Control design

Let us suppose that the disturbance $w(t)$ affecting system (1) satisfies (2), and the discontinuous term is such that $0 < \alpha \leq \alpha_c$, for some known bound α_c. The control objective is to find a control u, depending on x_1 and x_2, such that the closed-loop response of system (1)-(2) satisfies

$$x_1^2 + x_2^2 = r^2 \tag{4}$$

where r is the amplitude of the oscillation signal that we propose. Based on (4) two sliding surfaces are designed to have a sliding surface as in Figure 1

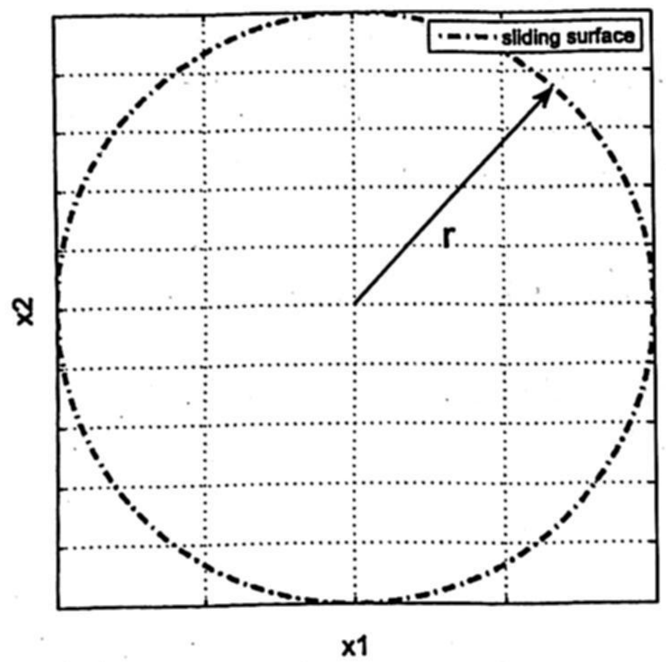

Fig. 1. Phase portrait of the proposed sliding surface.

$$s = \begin{cases} \sqrt{-x_1^2 + r} + x_2, & \text{if } x_2 \leq 0, \\ -\sqrt{-x_1^2 + r} + x_2, & \text{if } x_2 > 0. \end{cases} \tag{5}$$

The switching surface (5) can be represented as one under the following expression

$$s = -\frac{x_2}{|x_2|}\sqrt{-x_1^2 + r} + x_2. \tag{6}$$

A control law designed from (6), which can ensure us that trajectories (x_1, x_2) are going to converge to the sliding surface is given by

$$u = g(x)^{-1}\left(Ax_1 + Bx_2 + \frac{-x_1|x_2|}{\sqrt{-x_1^2 + r}} - \lambda s - \beta\,\text{sign}(s)\right). \tag{7}$$

where the parameters λ and β are positive; they will be tuned to ensure the motion of the trajectories be directed towards the sliding surface.

4 Stability analysis

We analyze in this section the stability of the closed-loop system (1), controlled by (7), and conclude about the overall stability.

By substituting (7) into (1), the closed-loop system takes the form

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \frac{-x_1|x_2|}{\sqrt{-x_1^2 + r}} - \lambda s - \beta\,\text{sign}(s) - \alpha\,\text{sign}(x_2) + w(t). \end{aligned} \tag{8}$$

Now, we ensure the existence of sliding modes by verifying $s\dot{s} < 0$. To this end, note that, from (2) and the fact that $\alpha \leq \alpha_c$, then

$$\begin{aligned} s\dot{s} &= s\left(\frac{-|x_2|+x_2\,\mathrm{sign}(x_2)+|x_2|^2}{|x_2|^2}\dot{x}_2 + \frac{x_1x_2}{\sqrt{-x_1^2+r|x_2|}}\dot{x}_1\right) \\ &= s\left(-\lambda s - \beta\,\mathrm{sign}(s) + w(t) - \alpha\,\mathrm{sign}(x_2)\right) \\ &\leq -\lambda s^2 - \beta|s| + (M+\alpha_c)\,|s| \\ &\leq -\lambda s^2 - (\beta - (M+\alpha_c))\,|s|. \end{aligned}$$

We conclude the existence of sliding modes on the surface (6) while the condition $\beta > M + \alpha_c$ be satisfied. This gives a guide to tune the parameter β of the controller (7). In fact, we can demonstrate that the trajectories reach the surface $s = 0$, in finite time, using the quadratic function

$$V(s) = s^2, \tag{9}$$

and compute its time derivative along the solutions of (8),

$$\begin{aligned} \dot{V}(s(t)) &\leq -2\lambda s^2 - 2(\beta - (M+\alpha_c))\,|s| \leq -2(\beta - (M+\alpha_c))\,|s| \\ &= -2(\beta - (M+\alpha_c))\sqrt{V(s(t))}. \end{aligned} \tag{10}$$

From (10) it follows that

$$V(t) = 0 \quad \text{for} \quad t \geq t_0 + \frac{\sqrt{V(t_0)}}{(\beta-(M+\alpha_c))} = t_f. \tag{11}$$

Hence, $V(t)$ converges to zero in finite time and, in consequence, a motion along the manifold $s = 0$ occurs in the discontinuous system (8). Notice that the reaching time can be reduced by increasing the value of parameter β.

5 Academic example

Performance issues and robustness properties of the proposed sliding mode controller have been tested with some numerical experiments under the following parameters as shown in Table 1

Table 1. Plant parameters, controller gains, amplitude of the discontinuous term, disturbances and initial conditions.

A	B	λ	β	r	α	w	$x_1(0)$	$x_2(0)$
65	7	340	5	1	1.5	$0.5\sin(10t)$	$\pi/4$	π

According to (1) the academic example takes the form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}x_2\\ -65x_1 - 7x_2 - 1.5\,\mathrm{sign}(x_2)\end{bmatrix} + \begin{bmatrix}0\\ u\end{bmatrix} + \begin{bmatrix}0\\ 0.5\sin(10t)\end{bmatrix} \tag{12}$$

The Figure 2 displays the system trajectories x_1 and x_2 which must converge to the proposed sliding surface as is shown in Figure 3. The control signal is shown in Figure 4, the chattering effect can be reduce by adjusting the β parameter, this can occur because β denotes the amplitude of the discontinuous control term, just keeping in mind that $\beta > M + \alpha_c$ must be satisfied. The sliding motion which converge to $s = 0$ in finite time approximately in $t = 0.2$ seconds is presented in Figure 5.

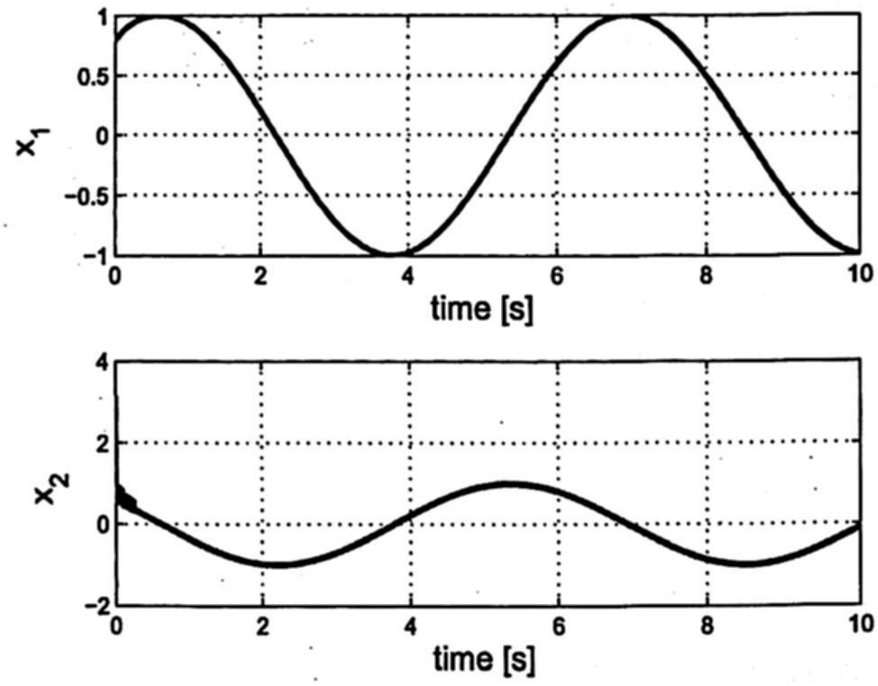

Fig. 2. Trajectories x_1 and x_2.

6 Conclusions

Has been introduced a way to generate a limit cycle in a nonlinear second order system through a sliding surface design, beside of circular phase portraits it can be propose ellipsoidal sliding surfaces by using the methodology mentioned afore, moreover it can be change the frequency of oscillation by adding a constant gain θ to the sliding surface as follows $s = -\frac{x_2}{|x_2|}\sqrt{-x_1^2 + r} + \theta x_2$. It is worth to mention that the proposed controller does not need an exact knowledge of the discontinuous function and disturbances; it only needs an upper bound of their magnitudes. It is proved that the limit cycle is reached in finite time in spite of the aforementioned uncertainties. This control technique can be directly applied to second order mechanical systems, like a mass-spring-damper system. As a future work the authors are interested in analyze the dynamical behavior of the trajectories once reached the sliding surface, also it would be of our interest to generate limit cycles with different geometrical shapes. It is important to point out that it has not been proved that the trajectories will remain in the sliding

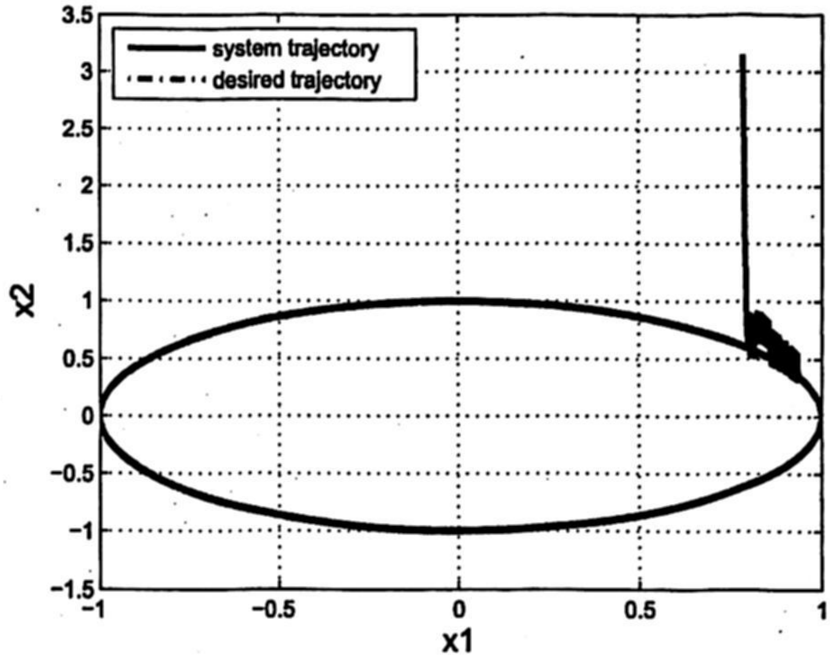

Fig. 3. Phase portrait, where can be seen the trajectories convergence to the proposed sliding surface.

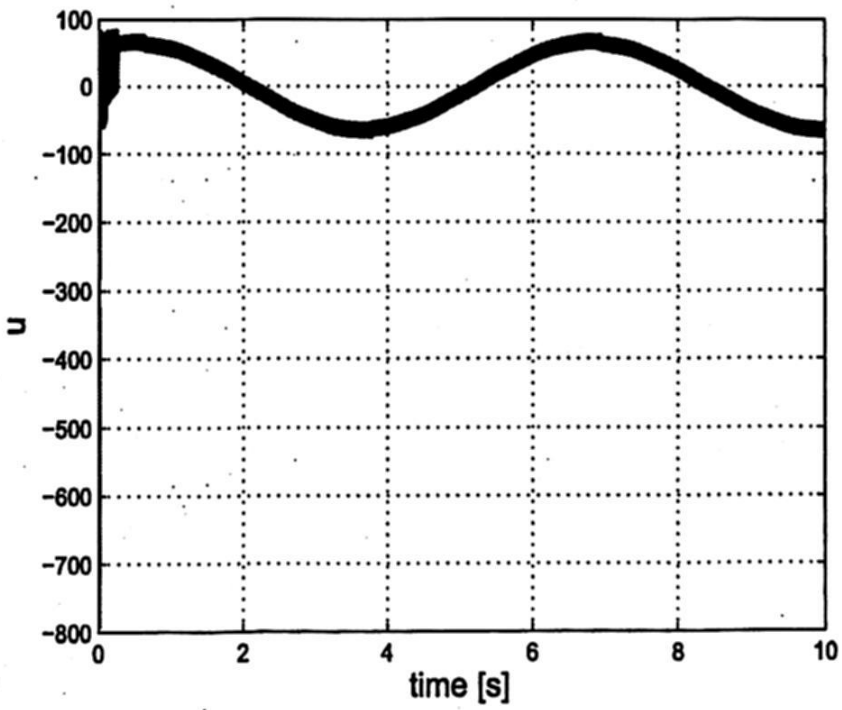

Fig. 4. Control signal, where the chattering phenomenon appears in steady state.

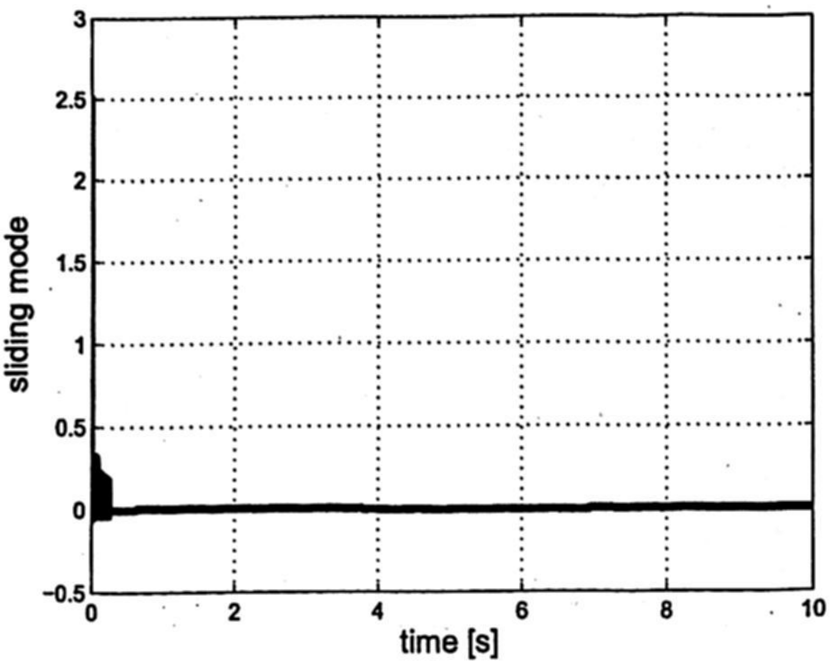

Fig. 5. Sliding motion, observe that the chattering phenomenon has a a relatively small amplitude.

surface for all time, so it is likely that trajectories can scape from the sliding surface at a time, further analysis should be done about this concern.

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